Probability Distributions of Stress Peaks in Linear and Nonlinear Structures

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The first part of this paper is a review of current knowledge of the probability distribution of stress peaks for a general type of broadband response of a linear structure under stationary and Gaussian random excitations. It is shown that the random stress in a linear structure always can be decomposed into two components and that the peaks in one of the components are Rayleigh-distributed. The second part of this paper presents a solution to the distribution of stress peaks in a nonlinear structure, with the restrictions that the response of the structure is dominated by a single mode and that the nonlinear deflection (not necessarily the stress) is of a narrow band. The theoretical result for the nonlinear stress peaks shows marked differences from that of the linear stress peaks. In particular, there is a finite probability measure for the stress peaks to assume negative values in a nonlinear structure, even when the deflection peaks are always positive. For the positive stress peaks, which are smaller than about two times the standard deviation of the stresses, the nonlinear theory predicts a smaller probability measure than that of a Rayleigh distribution.

Nomenclature

a, b = length and width of plate in the x and y directions, respectively
c = thickness of plate
C = constant (with or without subscript)
$E\{\ \}$ = statistical average
E = Young's modulus
M = matrix of second moments
M_s = expected number of peaks per unit time above level s
M_t = total expected number of peaks per unit time regardless of level
M_α = expected number of positive peaks per unit time above level $\alpha \geq 0$
$M_{\alpha-}$ = expected number of negative peaks per unit time above level $\alpha \leq 0$
$N_{\alpha+}$ = expected number of crossings at positive slopes at level α
p = probability density (subscript or subscripts indicate the random process or processes considered; probability density for peaks is denoted when subscript is omitted)
q = generalized displacement
r = a/b
s = stress
t = time
u, v, w = displacements in the x,y, and z directions, respectively
x, y, z = coordinate axes
$\{\ \}$ = random process
$\alpha,\beta,\gamma,\xi,\eta$ = range variables for random processes
δ = dimensionless nonlinear parameter
ϵ = nonlinear parameter
ϵ_x, ϵ_y = strains
ν = Poisson's ratio
σ^2 = second moments (with subscripts 1, 2, 3)
σ_0^2 = mean square linear deflection obtained by setting nonlinear parameter to zero
χ = normalized nonlinear stress
ω_0 = linear frequency
ζ = damping factor

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Introduction

THE knowledge of the probability distribution of stress (or strain)† peaks in a randomly excited structure is required in the estimation of fatigue life of the structure. For reasons of mathematical expediency and the nature of most forcing fields, the excitation generally is assumed to be Gaussian and stationary; then the induced stress in a linear structure also is Gaussian and approaching stationarity, for which the problem of peak distribution long has been solved[1] but sometimes is not well understood by many structural engineers whose duties are related to random fatigue. For this reason, the first part of this paper is devoted to a review of the problem of linear structures.

The second part of this paper takes up the question of stress peak distribution in a nonlinear structure. This problem is considerably more difficult, and certain limitations are imposed in order to obtain a solution. These restrictions are 1) the response of the structure is dominated by a single mode; 2) the excitation may be approximated by white noise in the frequency range of interest; and 3) the random deflection (not necessarily the stress) is of a narrow band. Admittedly, the combined restrictions represent a mathematical model that is highly idealized. The model may be quite unrealistic in some cases, whereas it may be acceptable in others. However, it is felt that a simple model best can provide the insight of the theoretical parameters involved, which will serve as a guide for future research.

Distribution of Peaks for an Arbitrary Random Process

Let M_s be the expected number of peaks above the level $y = s$ per unit time in a random process $\{y(t)\}$. The expected number of peaks between $y = s$ and $y = s + ds$ per unit time is given by

$$m_s = -(dM_s/ds)ds \tag{1}$$

The probability that a peak occurs between $y = s$ and $y =$

† Since the majority of the discussion to follow applies to both the stress and the strain, for simplicity only the stress will be referred to in those cases.

$s + ds$ is then

$$p(s)ds = \frac{m_s}{M_T} = -\frac{1}{M_T}\cdot\frac{dM_S}{ds}ds \qquad (2)$$

where M_T is the total expected number of peaks per unit time regardless of the magnitude of the peaks. Thus,

$$p(s) = -(1/M_T)\cdot(dM_s/ds) \qquad (3)$$

is the probability density of peaks in an arbitrary random process.

The general expressions for M_s and M_T involve the joint probability density of the following: 1) the random process considered, $\{y(t)\}$; 2) its first time derivative, $\{\dot{y}(t)\} = \{f(t)\}$; and 3) its second time derivative, $\{\ddot{y}(t)\} = \{g(t)\}$, assuming that both of the derivatives exist with probability one. This joint probability density will be denoted by $p_{yfg}(\alpha,\beta,\gamma;t)$. In this notation, the random processes $\{y(t)\}$, $\{f(t)\}$, and $\{g(t)\}$ may be nonstationary; however, these random processes are "sampled" at a same time instant t. Rice[1] has shown that

$$M_s(t) = \int_s^\infty d\alpha \int_{-\infty}^0 -\gamma\, p_{yfg}(\alpha,0,\gamma;t)d\gamma \qquad (4)$$

$$M_T(t) = \int_{-\infty}^\infty d\alpha \int_{-\infty}^0 -\gamma\, p_{yfg}(\alpha,0,\gamma;t)d\gamma \qquad (5)$$

If the random processes considered are stationary, then Eqs. (4) and (5) reduce to

$$M_s = \int_s^\infty d\alpha \int_{-\infty}^0 -\gamma\, p_{yfg}(\alpha,0,\gamma)d\gamma \qquad (6)$$

$$M_T = \int_{-\infty}^\infty d\alpha \int_{-\infty}^0 -\gamma\, p_{yfg}(\alpha,0,\gamma)d\gamma \qquad (7)$$

independent of time. Thus the random process of peaks, which is characterized by $p(s)$, is stationary or nonstationary depending upon whether or not $\{y(t)\}$, $\{f(t)\}$, and $\{g(t)\}$ are stationary.

Stress Peaks in a Linear Structure

It will be assumed that the excitation applied to the structure is stationary and Gaussian. The response of the structure, being a linear function of the excitation, also must be Gaussian, and it becomes stationary in a short time after the transient motion is damped out. In the present sense, the response of the structure may be either the deflection or the stress or the strain at a certain point on the structure. Since these quantities are all related linearly, there will be no essential difference in the theoretical treatment for one from the others.

For simplicity, the statistical mean of the excitation and, hence, that of the steady state (in a statistical sense) response are assumed to be zero. The joint probability density $p_{yfg}(\alpha,\beta,\gamma)$, which is now time-independent and Gaussian, is determined completely from various second moments. Rice has shown that the joint probability density that is required in the expression for M_s is given by

$$p_{yfg}(\alpha,0,\gamma) = (2\pi)^{-3/2}|\mathbf{M}|^{-1/2}\exp[-\tfrac{1}{2}|\mathbf{M}|^{-1}\cdot(\sigma_2^2\sigma_3^2\alpha^2 + 2\sigma_2^4\alpha\gamma + \sigma_1^2\sigma_2^2\gamma)] \qquad (8)$$

where, in the present notation,

$$\sigma_1^2 = E\{y^2\}$$
$$\sigma_2^2 = E\{f^2\} \qquad (9)$$
$$\sigma_3^2 = E\{g^2\}$$

and

$$|\mathbf{M}| = \sigma_2^2(\sigma_1^2\sigma_3^2 - \sigma_2^4) \qquad (10)$$

is the determinant of the matrix of the second moments. It can be shown[2] that $|\mathbf{M}| > 0$; thus $p_{yfg}(\alpha,0,\gamma)$ is well defined. In the derivation of Eq. (8), Rice has assumed that the random processes are ergodic and that the various second moments may be computed from the time averages. Substitution of (8) into (6) and (7) results in

$$M_s = \int_s^\infty (2\pi)^{-3/2}(\sigma_1\sigma_2)^{-2}\left\{|\mathbf{M}|^{1/2}\exp\left(\frac{-\sigma_2^2\sigma_3^2\alpha^2}{2|\mathbf{M}|}\right) + \sigma_2^4\alpha\left(\frac{\pi}{2\sigma_1^2\sigma_2^2}\right)^{1/2}\left[1 + \operatorname{erf}\left(\frac{\sigma_2^3\alpha}{2|\mathbf{M}|^{1/2}\sigma_1}\right)\right]\cdot \exp\left(-\frac{\alpha^2}{2\sigma_1^2}\right)\right\}d\alpha \qquad (11)$$

where erf$x$ is the well-known error function defined by

$$\operatorname{erf}x = \frac{1}{(2\pi)^{1/2}}\int_{-x}^x \exp\left(-\frac{u^2}{2}\right)du = \frac{2}{\pi^{1/2}}\int_0^x \exp(-t^2)dt$$

and

$$M_T = (1/2\pi)(\sigma_3/\sigma_2) \qquad (12)$$

From Eq. (3)

$$p(s) = (2\pi)^{-1/2}(\sigma_1^2\sigma_2\sigma_3)^{-1}\times\left\{|\mathbf{M}|^{1/2}\exp\left(\frac{-\sigma_2^2\sigma_3^2s^2}{2|\mathbf{M}|}\right) + \sigma_2^4 s\left(\frac{\pi}{2\sigma_1^2\sigma_2^2}\right)^{1/2}\times \left[1 + \operatorname{erf}\left(\frac{\sigma_2^3 s}{2^{1/2}|\mathbf{M}|^{1/2}\sigma_1}\right)\right]\cdot\exp\left(-\frac{s^2}{2\sigma_1^2}\right)\right\} \qquad (13)$$

Equation (13) may be written alternatively as follows:

$$p(s) = (2\pi)^{-1/2}\sigma_1^{-1}\left[1 - \left(\frac{N_{0+}}{M_T}\right)^2\right]^{1/2}\cdot \exp\left\{\frac{-s^2}{2\sigma_1^2}\left[1 - \left(\frac{N_{0+}}{M_T}\right)^2\right]^{-1}\right\} + \frac{1}{2\sigma_1}\left(\frac{N_{0+}}{M_T}\right)\left(\frac{s}{\sigma_1}\right)\cdot \left\{1 + \operatorname{erf}\left[\frac{s}{2^{1/2}\sigma_1}\left(\frac{M_T^2}{N_{0+}^2} - 1\right)^{-1/2}\right]\right\}\exp\left(\frac{-s^2}{2\sigma_1^2}\right) \qquad (14)$$

where $N_{0+} = (2\pi)^{-1}(\sigma_2/\sigma_1)$ is the number of times per unit time that $y(t)$ crosses the line $y = 0$ at positive slopes. This alternative form written in the present notation is due to Huston and Skopinski.[3]

The probability distribution of the response peaks as given in Eq. (14) has been plotted in Ref. 3 for various N_{0+}/M_T values. It has been pointed out[4] that N_{0+}/M_T must lie within the interval (0,1). The upper limit $N_{0+}/M_T = 1$ corresponds to a special case illustrated in Fig. 1, where the expected number of peaks is equal to the expected number of zero crossings at positive (or negative) slopes. For this special

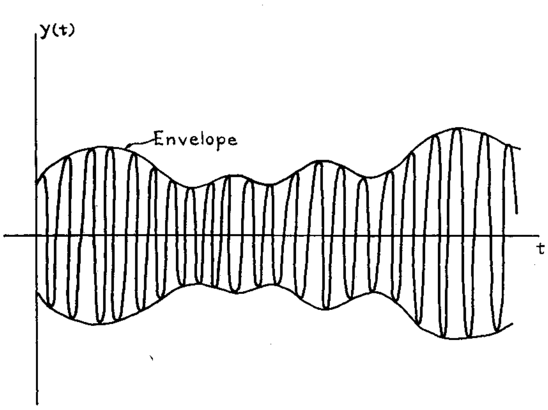

Fig. 1 Typical record of narrow band random stresses; number of peaks equal number of zero crossings at positive (or negative) slopes

case, Eq. (14) reduces to

$$p(s) = \frac{1}{\sigma_1}\left(\frac{s}{\sigma_1}\right)\exp\left(\frac{-s^2}{2\sigma_1^2}\right) \tag{15}$$

representing a Rayleigh distribution. On the other hand, when N_{0+}/M_T is very small, corresponding to a situation where the expected number of maxima is much larger than the expected number of zero crossings, Eq. (14) approaches in the limit

$$p(s) = \frac{1}{(2\pi)^{1/2}\sigma_1}\exp\left(-\frac{s^2}{2\sigma_1^2}\right) \tag{16}$$

for a Gaussian distribution. Figure 2 shows a typical stress record (in solid line) with a relatively small N_{0+} to M_T ratio. Other than these extreme cases, the expression for the probability density, Eq. (14), is neither Rayleigh nor Gaussian.

The result given in Eq. (15) agrees with the well-known fact that the envelope of a stationary ergodic Gaussian random process follows a Rayleigh distribution if the process is of a narrow band.‡ The narrowness of the bandwidth insures that, in a typical record of the random process, the number of peaks is nearly equal to the number of zero crossings at positive (or negative) slopes. Furthermore, the probability distribution of the peaks is approximately the same as that of the envelope. In the case of a wide band random process, the behavior of the envelope has not been studied, possibly because there has not been a generally agreed definition for such an envelope. It can be shown, however, that a stationary ergodic Gaussian process, regardless of the bandwidth, always can be decomposed into two components with the peaks in one of the components Rayleigh-distributed.

Consider each of the random records in an ensemble, such as that shown in Fig. 2. Subject to certain restrictions (to be discussed), a curve (in dotted line) is drawn passing through an arbitrary point between every peak and the following trough. The ensemble of all the dotted curves so obtained represents a new random process that will be denoted by $\{y_1(t)\}$. The difference between the original random process $\{y(t)\}$ and $\{y_1(t)\}$ is another new random process given by

$$\{y_2(t)\} = \{y(t) - y_1(t)\} \tag{17}$$

It has been assumed previously that $\{y(t)\}$ is Gaussian. In obtaining $\{y_1(t)\}$, the following restrictions will be made:

1) $\{y_1(t)\}$ is Gaussian.

2) $\{y_2(t)\} = \{y(t) - y_1(t)\}$ is stationary and ergodic. This restriction includes the special case in which $\{y(t)\}$ and $\{y_1(t)\}$ are both stationary and ergodic.

3) The time coordinates of the peaks in each member of $\{y_2(t)\}$ coincide with those of the corresponding member in $\{y(t)\}$.

These restrictions insure that $\{y_2(t)\}$ is a stationary ergodic Gaussian random process that satisfies $N_{0+}/M_T = 1$. Thus the distribution of peaks in $\{y_2(t)\}$ must follow Eq. (15).

Recently, Schjelderup and Galef[5] have studied a number of strain records that were taken from actual structures subjected to acoustic noise and were supplied to them by several leading airplane companies. Records from both linear and nonlinear structures were considered. The main purpose of their study was to determine experimentally the probability distribution of strain peaks as a preliminary step toward prevention of acoustically induced fatigue. The results published in Ref. 5 have shown that the alternating strain follows approximately a Rayleigh distribution, where the alternating strain is defined as $\frac{1}{2}$ of the difference between a strain peak and the following strain trough. These results since have attracted attention both in this country and abroad. Clarkson,[6] in a lecture delivered at the Royal Aeronautical Society in England, has suggested using an S-N curve determined from Rayleigh-distributed random stresses for the design of structures to resist jet noise fatigue.

‡ For proof see, for example, Ref. 2, pp. 134–136.

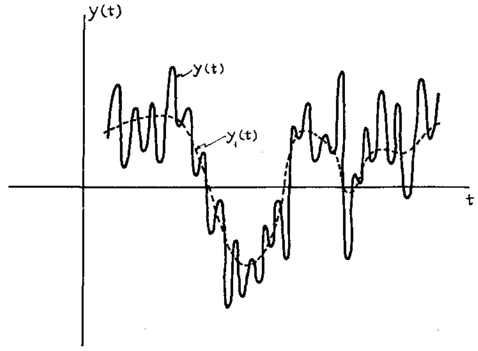

Fig. 2 Typical record of random stresses with number of peaks greater than number of zero crossings at positive (or negative) slopes; dotted curve shows the proposed decomposition of such a record

The problem of Schjelderup and Galef is a special case of $\{y_2(t)\}$ discussed in the foregoing. The special case further requires that the dotted curve in Fig. 2 pass through the midpoint between every peak and the following trough. It has been assumed that the original random process $\{y(t)\}$ is Gaussian, stationary, and ergodic. It remains to substantiate that $\{y_1(t)\}$ obtained in the manner suggested in Ref. 5 also is Gaussian, stationary, and ergodic. It is plausible to postulate that $\{y_1(t)\}$ represents essentially the lower harmonic elements of $\{y(t)\}$; hence it must be Gaussian, stationary, and ergodic if one uses Rice's representation of a stationary, ergodic, Gaussian process in Fourier expansions and argues from the fact that the Fourier coefficients are independent Gaussian random variables. Therefore, the theoretical prediction agrees with the results obtained in Ref. 5 in the case of linear structures.

It is interesting to note that the decomposition suggested in Ref. 5 is not the only way by which one can obtain a component random process $\{y_2(t)\}$ that satisfies the condition $N_{0+}/M_T = 1$. The rules to be observed in such a decomposition have been stated clearly in restrictions 1–3 following Eq. (17) which are much weaker than requiring that the dotted curve in Fig. 2 pass through the midpoint between every peak and the following trough.

Stress Peaks in a Nonlinear Structure

At the present time, the response of a nonlinear structure under random excitation can be determined analytically only when the motion of the structure is dominated by a single mode.§ The equation governing the generalized displacement $q(t)$ of a single-degree-of-freedom nonlinear system may be written as follows:[8]

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2[q + \epsilon g(q)] = \psi(t) \tag{18}$$

where ζ is the damping factor, ω_0 and ϵ are constants determined from the dimensions and material properties of the structure, and $g(q)$ is an odd single-valued function with potential

$$G(q) = \int_0^q g(q')dq' \tag{19}$$

Let $\{\psi(t)\}$ be a Gaussian white noise, and consider the generalized displacement $\{q\}$ and the velocity $\{\dot{q}\} = \{h(t)\}$ as a two-dimensional Markoff process. The solution to Eq.

§ Multimode solutions are available for nonlinear strings (see, for example, Ref. 7), which are of theoretical interest. However, a string hardly can be considered as a practical structure.

(18) in the stationary state is represented by the joint probability density:

$$p_{qh}(\xi, \eta) = C \exp\left\{-\frac{\eta^2}{2\sigma_0^2\omega_0^2} - \frac{\xi^2}{2\sigma_0^2} - \epsilon\frac{G(\xi)}{\sigma_0^2}\right\} \quad (20)$$

where σ_0^2 is the mean-square linear deflection obtained by setting the nonlinearity parameter to zero, and C is the normalization factor determined from the condition

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{qh}(\xi,\eta)d\xi d\eta = 1$$

It is interesting to note from Eq. (20) that the distribution of the velocity $\{h(t)\}$ is Gaussian, whereas the distribution of the displacement $\{q(t)\}$ is not, and that $\{q(t_i)\}$ and $\{h(t_i)\}$ are independent random variables when sampled at the same time t_i, a direct consequence of the assumption of stationarity. Since the motion of the structure is dominated by a single mode, $\{q(t)\}$ is expected to be of a narrow band for small damping.‖ For a narrow band random process, almost all the peaks are positive. A typical record of $\{q(t)\}$ will have a general appearance as shown in Fig. 1, although it has been shown that $\{q(t)\}$ is non-Gaussian.

To establish the relation between the generalized displacement and the stress at a given point in a structure, the example of a flat plate simply supported along $x = 0, a$ and $y = 0, b$ is considered. The stress-strain relation of the plate material is assumed to remain linear, thus the nonlinearity is due solely to the stretching of the plate when the deflection is not small in comparison with the plate thickness. Chu and Herrmann[9] have shown that, if the fundamental bending mode is predominant, then the motion of the plate can be represented adequately as follows:

$$u = \frac{\pi q^2}{16a}\left[\cos\left(\frac{2\pi y}{b}\right) - 1 + \nu r^2\right]\sin\left(\frac{2\pi x}{a}\right)$$

$$v = \frac{\pi q^2}{16b}\left[r\cos\left(\frac{2\pi x}{a}\right) - r + \frac{\nu}{r}\right]\sin\left(\frac{2\pi y}{b}\right) \quad (21)$$

$$w = q\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

The tensile strains on the surface of the plate are given by

$$\epsilon_x = (v_{,x} + \tfrac{1}{2}w_{,x}^2) + [(c/2)w_{,xx}]$$
$$\epsilon_y = (u_{,y} + \tfrac{1}{2}w_{,y}^2) + [(c/2)w_{,yy}] \quad (22)$$

where a comma preceeding a subscript (subscripts) indicates partial differentiation (differentiations). The corresponding tensile stresses are given by

$$s_x = [E/(1-\nu^2)](\epsilon_x + \nu\epsilon_y)$$
$$s_y = [E/(1-\nu^2)](\epsilon_y + \nu\epsilon_x) \quad (23)$$

From Eqs. (21–23), a general expression is obtained for the stress (or strain)# at any given point in the structure:

$$s = C_1 q + C_2 q^2 \quad (24)$$

where C_1 and C_2 are constants. It can be shown that Eq. (24) is valid also for shearing stress or stress in any direction.

Depending on the type of stress considered, the constants C_1 and C_2 in Eq. (24) can be determined from the material properties and dimensions of the plate and the location and direction at which the stress is to be measured. For tensile stress,

‖ Every condition being the same, the bandwidth of a nonlinear deflection may be somewhat wider than that corresponding to linear deflection.

Since the majority of the discussion applies to both the stress and the strain, for simplicity only the stress will be referred to in those cases.

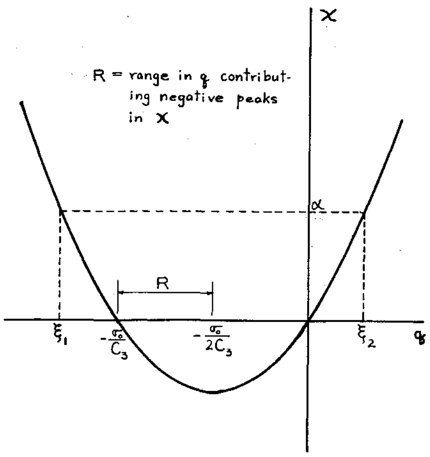

Fig. 3 Relation between the nonlinear deflection q and the nonlinear stress χ

C_2 always must be positive, whereas C_1 may be taken as positive without loss of generality by choosing an appropriate positive direction for q. In fact, Eq. (24) may be normalized by dividing both sides by C_1, resulting in a simplified expression

$$\chi = s/C_1 = q + C_3(q^2/\sigma_0) \quad (25)$$

where C_3 is a positive nondimensional constant. Figure 3 shows the relationship between χ and q.

It is not possible at the present time to find the joint probability density for the normalized stress $\{\chi\}$, its first time derivative $\{\dot{\chi}\} = \{\kappa(t)\}$, and its second time derivative $\{\ddot{\chi}\} = \{\mu(t)\}$ required in determining the distribution of peaks of the stress $\{\chi\}$ for the most general case. However, if $\{q\}$ is of a narrow band, then nearly all peaks in $\{q\}$ are positive, and all troughs in $\{q\}$ are negative. As a result, there will be almost no positive troughs in $\{\chi\}$, although there may exist negative peaks in $\{\chi\}$. It can be shown that each negative peak in $\{\chi\}$ is caused by a negative trough in $\{q\}$ between $q = -\sigma_0/C_3$ and $q = -\sigma_0/(2C_3)$, as indicated in Fig. 3. The former corresponds to the maximum negative peak of $\chi(=0)$ and the latter to the minimum negative peak of χ $[= -\sigma_0/(4C_3)]$. The formation of a negative stress peak is illustrated in Fig. 4. In what follows, it is more convenient to handle the positive peaks and the negative peaks in $\{\chi\}$ separately.

Since there is almost no positive trough in $\{\chi\}$, the positive peaks in $\{\chi\}$ above a given level, $\chi = \alpha \geq 0$, are almost equal to the number of crossings at positive (or negative) slopes at that level. Thus,[1]

$$M_\alpha \approx N_{\alpha+} = \int_0^\infty \beta p_{\chi\kappa}(\alpha,\beta)d\beta \quad (26)$$

The joint probability density $p_{\chi\kappa}(\alpha,\beta)$ required in the application of Eq. (26) may be determined in the following way: from the known joint probability density $p_{qh}(\xi,\eta)$, one first finds the joint density, $p_{q\kappa}(\xi,\beta)$, using the relationship

$$\dot{\chi} = \dot{q}[1 + (2C_3/\sigma_0)q]$$
or
$$\kappa = h[1 + (2C_3/\sigma_0)q] \quad (27)$$

Equation (27), which is obtained from differentiation of Eq. (25) with respect to time, shows that κ is a monotone function of h for a given q. It follows from the well-known theorem for transformation between monotonic random variables that

$$p_{q\kappa}(\xi,\beta) = p_{qh}\left(\xi, \eta = \frac{\beta}{1 + (2C_3/\sigma_0)\xi}\right)\left|\frac{1}{1 + (2C_3/\sigma_0)\xi}\right| \quad (28)$$

The required $p_{\chi\kappa}(\alpha,\beta)$ then is determined from

$$p_{\chi\kappa}(\alpha,\beta) = \frac{\partial}{\partial\alpha}\int_{\xi_1}^{\xi_2} p_{q\kappa}(\xi,\beta)d\xi \quad (29)$$

where

$$\xi_{1,2} = \frac{-1 \mp [1 + 4C_3(\alpha/\sigma_0)]^{1/2}}{2}\frac{\sigma_0}{C_3} \quad (30)$$

are the roots of q obtained from Eq. (25) by letting $\chi = \alpha$. Using (20) in (28) and substituting the result in (29), one obtains

$$p_{\chi\kappa}(\alpha,\beta) = \frac{C}{1 + 4C_3(\alpha/\sigma_0)} \exp\left\{-\frac{\beta^2}{2\sigma_0^2\omega_0[1 + 4C_3(\alpha/\sigma_0)]}\right\}\times \left\{\exp\left[-\frac{\xi_2^2}{2\sigma_0^2} - \delta\frac{G(\xi_2)}{\sigma_0^4}\right] + \exp\left[-\frac{\xi_1^2}{2\sigma_0^2} - \delta\frac{G(\xi_1)}{\sigma_0^4}\right]\right\} \quad (31)$$

where $\delta = \epsilon\sigma_0^2$ is a nondimensional nonlinearity parameter. Substitution of (31) into (26) results in

$$M_\alpha = C\sigma_0^2\omega_0^2\left\{\exp\left[-\frac{\xi_2^2}{2\sigma_0^2} - \frac{\delta}{\sigma_0^4}G(\xi_2)\right] + \exp\left[-\frac{\xi_1^2}{2\sigma_0^2} - \frac{\delta}{\sigma_0^4}G(\xi_1)\right]\right\} \quad (32)$$

Attention now will be directed to negative peaks in $\{\chi\}$. It is observed that the number of negative peaks in $\{\chi\}$ above $\chi = \alpha \leq 0$ is equal to the number of troughs in $\{q\}$ between $q = -\sigma_0/C_3$ and $q = \xi_1$, where ξ_1 is given in Eq. (30). Since $p_{qh}(\xi,\eta)$ is an even function of ξ, it follows that positive and negative q are distributed identically. Thus, the expected number of negative peaks in $\{\chi\}$ above $\chi = \alpha \leqq 0$ per unit time is

$M_{\alpha-}$ = expected number of troughs in q in the interval $[(-\sigma_0/C_3), \xi_1]$

= expected number of peaks in q in the interval $[-\xi_1, (\sigma_0/C_3)]$

$$= \left[\int_0^\infty \eta p_{qh}(\xi,\eta)d\eta\right]_{\xi = -\xi_1} - \left[\int_0^\infty \eta p_{qh}(\xi,\eta)d\eta\right]_{\xi=\sigma_0/C_3}, \quad -\sigma_0/(4C_3) \leq \alpha \leq 0 \quad (33)$$

Using (29) in (33), one obtains

$$M_{\alpha-} = C\sigma_0^2\omega_0^2\left\{\exp\left[-\frac{(-\xi_1)^2}{2\sigma_0^2} - \delta\frac{G(-\xi_1)}{\sigma_0^4}\right] - \exp\left[-\frac{1}{2C_3^2} - \delta\frac{G(\sigma_0/C_3)}{\sigma_0^4}\right]\right\}, \quad -\sigma_0/(4C_3) \leq \alpha \leq 0 \quad (34)$$

The total expected number of peaks in χ is computed from

$$M_T = (M_\alpha)_{\alpha=0} + (M_{\alpha-})_{\alpha=-\sigma_0/(4C_3)} \quad (35)$$
$$= C\sigma_0^2\omega_0^2\{1 + \exp[-(8C_3^2)^{-1} - \delta\sigma_0\ {}^4G(\sigma_0/2C_3)]\}$$

The probability density for the distribution of nonlinear stress peaks now is obtained by straightforward substitution into Eq. (3), giving

$$p(\alpha) = \left\{1 + \exp\left[-(8C_3^2)^{-1} - \delta\sigma_0^{-4}G\left(\frac{\sigma_0}{2C_3}\right)\right]\right\}^{-1}\cdot \left\{\left[\frac{\xi_2}{\sigma_0^2} + \frac{\delta}{\sigma_0^4}g(\xi_2)\right]\exp\left[-\frac{\xi_2^2}{2\sigma_0^2} - \frac{\delta}{\sigma_0^4}G(\xi_2)\right] - \left[\frac{\xi_1}{\sigma_0^2} + \frac{\delta}{\sigma_0^4}g(\xi_1)\right]\exp\left[-\frac{\xi_1^2}{2\sigma_0^2} - \frac{\delta}{\sigma_0^4}G(\xi_1)\right]\right\}\cdot \left\{1 + 4C_3\left(\frac{\alpha}{\sigma_0}\right)\right\}^{-1/2}, \quad \alpha \geq 0 \quad (36)$$

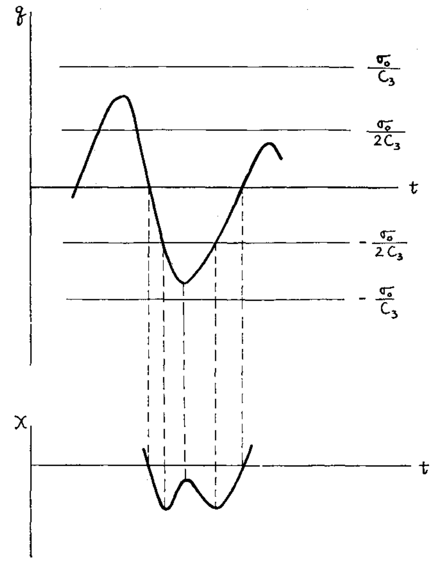

Fig. 4 Formation of negative stress peaks in a nonlinear structure

$$p(\alpha) = -\left\{1 + \exp\left[-(8C_3^2)^{-1} - \delta\sigma_0^{-4}G\left(\frac{\sigma_0}{2C_3}\right)\right]\right\}^{-1}\cdot \left[\frac{\xi_1}{\sigma_0^2} + \frac{\delta}{\sigma_0^4}g(\xi_1)\right]\exp\left[-\frac{\xi_1^2}{2\sigma_0^2} - \frac{\delta}{\sigma_0^4}G(\xi_1)\right]\cdot \left\{1 + 4C_3\left(\frac{\alpha}{\sigma_0}\right)\right\}^{-1/2}, \quad -\sigma_0/(4C_3) \leq \alpha \leq 0 \quad (37)$$

It is interesting to note that $p(\alpha)$ as given in Eqs. (36) and (37) is continuous at $\alpha = 0$; however, the slope of $p(\alpha)$ generally is discontinuous at $\alpha = 0$. For illustration, this probability density is plotted in Fig. 5 for 1) $C_3 = 0.5$, $\delta = 1$ and 2) $C_3 = 0.5$, $\delta = 1.5$. The first combination corresponds to the stress on the surface at the center of a nearly square flat plate simply supported on the four edges with a deflection of the same order of magnitude as the plate thickness. The second combination represents the same conditions as those stated in 1 except that the length to width ratio of the plate is assumed to be about 2. The plots in Fig. 5 show marked differences from the Rayleigh distribution, especially in the regions of negative stress peaks and positive stress peaks smaller than $2\sigma_0$. Since the probability distribution of the

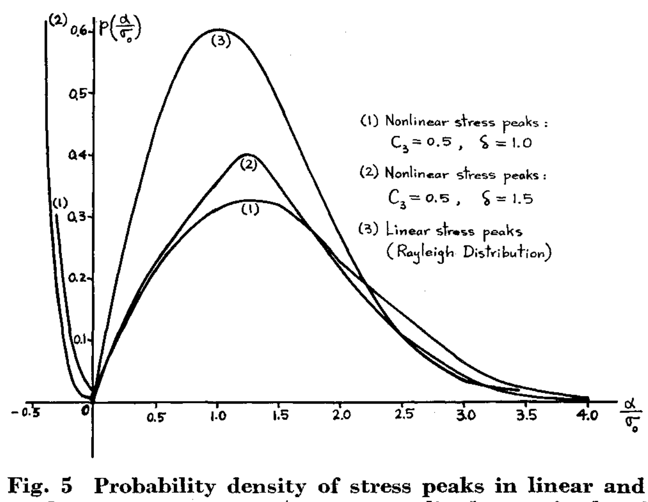

Fig. 5 Probability density of stress peaks in linear and nonlinear structures; α/σ_0 = normalized magnitude of stress peak; C_3, δ = constants depending on degree of nonlinearity

nonlinear stress is non-Gaussian, as can be seen from Eq. (31), for the alternating stress to be Rayleigh-distributed, as was claimed in Ref. 5, it is necessary that the non-Gaussian portion of the nonlinear stress is subtracted out exactly in the decomposition. It appears difficult to resolve such a conjecture. The experimental evidence provided in Ref. 5 that the alternating stress may be Rayleigh-distributed is considered inadequate in the present case, at least for smaller stress peaks (the questionable portion, as seen in Fig. 5). These smaller stress peaks have been neglected in truncation in Ref. 5.

Concluding Remarks

With the assumption that the random excitation is stationary and Gaussian, the stationary stress at any given location on a linear structure always can be decomposed into two components that are stationary and Gaussian, such that the peaks in one of the components are truly Rayleigh-distributed. This conclusion agrees with the experimental result recently obtained by Schelderup and Galef. On the other hand, analytically determined distribution of stress peaks in a nonlinear structure shows marked differences from that in a linear structure, especially in the regions of negative stress peaks and positive stress peaks smaller than $2\sigma_0$. Since previous experimental efforts have neglected the range of stress peaks, which is probably most questionable, it is suggested that further research be carried out with special attention to nonlinear response.

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